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LETTER TO THE EDITOR

Renormalization group analysis of superdiffusion in random velocity fields

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Abstract. Field-theoretic renormalization group theory is applied to the analysis of the asymptotic behaviour of diffusion in random velocity fields, each non-vanishing component of which is independent of the coordinate along the direction of that component. Below the critical dimension $d_c = 3$, the corresponding field theory is shown to have a non-trivial infrared stable fixed point of the renormalization group, which controls the long-time, large-distance behaviour of the model. The anomalous dimension ν has been found exactly in the $\varepsilon = 3 - d$ expansion, and the leading logarithmic corrections to the normal diffusion at $d = 3$ have been determined for all d': $1 \leqslant d' \leqslant d$, where d' is the number of components of the drift field.

Asymptotic behaviour of random walks in random media depends heavily both on the character of the effective driving random velocity field and on the range of its correlations: the potential part of the field hinders diffusion, whereas the divergenceless part enhances it, and the values of the corresponding critical exponents and dimensions depend on the range of the correlations [1,2]. In this letter we consider a family of recently proposed [3,4] models of diffusion in a divergenceless random field of a special spatial structure, which exhibit *superdiffusive* behaviour, i.e. the mean-square displacement of the random walks grows faster than linearly with time. The original model is connected with the description of ground water transport in heterogeneous rocks [3]. The basic feature of the unidirectional velocity field in this model is that it is independent of the coordinate along the field direction. The properties of diffusion in this unidirectional convection model and its isotropic generalization ('Manhattan grid' convection) have been recently studied both analytically and numerically [4].

In this letter, the problem is considered in a field-theoretic framework, in which the earlier results for the critical exponent ν of the mean-square displacement are confirmed and generalized. The critical dimension is confirmed to be $d_c = 3$, the critical exponent ν is determined at arbitrary dimensionality for both the unidirectional and isotropic Manhattan convection, and the corresponding results are also obtained for the intermediate case of $(d' < d)$-dimensional Manhattan convection in a d-dimensional space.

Consider a d-dimensional continuum system with stationary random velocity field $\boldsymbol{F}$ in x-direction: $\boldsymbol{F} = \boldsymbol{e}_x \psi(\boldsymbol{y})$, where the function ψ is a function of the transverse $\boldsymbol{y}$ coordinate only. The probability distribution $P(t, x, \boldsymbol{y})$ of a tracer particle at $(x, \boldsymbol{y})$

in a fixed field obeys the following diffusion equation

$$\left[\frac{\partial}{\partial t} - D_0^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - D_0^{\mathrm{L}}\frac{\partial^2}{\partial x^2} - \psi(y)\frac{\partial}{\partial x}\right]P(t,x,y) \equiv L_\psi P(t,x,y) = 0 \tag{1}$$

where D_0^{L} and D_0^{T} are the bare (unrenormalized) diffusion coefficients in the longitudinal and transverse directions, respectively. The random field ψ is assumed to have a Gaussian distribution with zero mean and the correlation function

$$\langle\psi(y)\psi(y')\rangle = \lambda_0\delta(y-y') \tag{2}$$

Here, the (non-negative) bare coupling constant λ_0 describes the strength of the disorder.

We shall calculate the Green function of the equation (1), averaged over the random field ψ. This stochastic problem may be cast into a field-theoretic form by the use of the functional-integral representation of the Green function

$$G_\psi(t-t',x-x',y,y') = \int \mathrm{D}\varphi\,\mathrm{D}\tilde\varphi\;\varphi(t,x,y)\tilde\varphi(t',x',y')\exp\left[\int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\tilde\varphi L_\psi\varphi\right].$$

Averaging over the distribution (2) we arrive at a functional integral over three fields φ, $\tilde\varphi$, and ψ for the $\varphi\tilde\varphi$ Green function $G_0 = \langle G_\psi\rangle$ of the field theory with the 'action'

$$S = -\frac{1}{2\lambda_0}\int \mathrm{d}y\,\psi^2 + \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\tilde\varphi\left[\frac{\partial}{\partial t} - D_0^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - D_0^{\mathrm{L}}\frac{\partial^2}{\partial x^2} - \psi(y)\frac{\partial}{\partial x}\right]\varphi. \tag{3}$$

It should be noted that since the field ψ does not depend on the variables t and x, there are no integrals over the Laplace variable and the momentum corresponding to the longitudinal coordinate x in the Laplace-momentum representation of the diagrammatic expansion of the Green function G_0. To determine the upper critical dimension of the field theory (3), we introduce for each variable v three scaling dimensions d_v^{s}, d_v^{L}, and d_v^{T} corresponding to the time, longitudinal and transverse coordinates, respectively. The scaling dimensions of all variables are determined from the condition that the action (3) is dimensionless with respect to time, longitudinal, and transverse coordinates separately. Ultimately, we are interested in the scale transformation, in which the bare propagator g_0 of the field theory (3) in the Laplace-momentum representation $g_0(s,k,p) = 1/(s + D_0^{\mathrm{L}}k^2 + D_0^{\mathrm{T}}p^2)$ is a homogeneous function of order -2 of its Laplace and momentum arguments:

$$g_0(\Lambda^2 s,\Lambda k,\Lambda p) = \Lambda^{-2}g_0(s,k,p). \tag{4}$$

The total scaling dimension of a variable v in such a scale transformation is therefore $d_v \equiv 2d_v^{\mathrm{s}} + d_v^{\mathrm{L}} + d_v^{\mathrm{T}}$. For example, for the diffusion coefficients we obtain $d_{D^{\mathrm{T}}}^{\mathrm{s}} = d_{D^{\mathrm{L}}}^{\mathrm{s}} = 1$, $d_{D^{\mathrm{T}}}^{\mathrm{L}} = d_{D^{\mathrm{L}}}^{\mathrm{T}} = 0$, and $d_{D^{\mathrm{L}}}^{\mathrm{L}} = d_{D^{\mathrm{T}}}^{\mathrm{T}} = -2$, which yield for the total dimensions the value $d_{D^{\mathrm{T}}} = d_{D^{\mathrm{L}}} = 0$. The scaling dimensions of bare and renormalized diffusion coefficients are the same; we have therefore omitted the subscript '0' in the preceding formulae. This is not so in the case of the coupling constant;

therefore the subscript must be retained. For the coupling constant λ_0 we obtain $d^{s}_{\lambda_0} = 2$, $d^{L}_{\lambda_0} = -2$, and $d^{T}_{\lambda_0} = (1-d)$, thus $d_{\lambda_0} = (3-d)$, from which it follows that the critical dimension of the model is $d_c = 3$.

Power counting in the graphs shows that the field theory (3) at three dimensions is not only renormalizable, but even *superrenormalizable*, i.e. it possesses only a finite number of superficially divergent graphs. The reason is that all the momentum factors generated by the derivative at the interaction vertex of the action (3) are factorized in the graphs of the model, since there are no momentum integrals over the longitudinal momentum variable. These momentum factors are, however, taken into account in determining the critical dimension of the model; therefore it is obvious that the higher the order of a particular graph, the faster the integrand of the graph must fall off at large transverse momenta.

In fact, there is only one superficially divergent graph left in the present model, the one-loop self-energy graph. Hence, in the minimal subtraction scheme there is only one renormalization constant Z, and the renormalized action may be written in the form

$$S_R = -\frac{1}{2\lambda\mu^\varepsilon}\int \mathrm{d}y\,\psi^2 + \int \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\bar{\varphi}\left[\frac{\partial}{\partial t} - D^{\mathrm{T}}\frac{\partial^2}{\partial y^2} - ZD^{\mathrm{L}}\frac{\partial^2}{\partial x^2} - \psi(y)\frac{\partial}{\partial x}\right]\varphi \tag{5}$$

where we have introduced renormalized diffusion constants D^{L} and D^{T}, renormalized coupling constant λ, and a scale-setting parameter μ of dimension of transverse momentum in order to make the renormalized coupling constant dimensionless under the scale transformation (4). As usual, $\varepsilon = d_c - d = 3 - d$. In general, the parameters λ and D^{T} differ from their bare counterparts at most by a finite renormalization, and in the minimal subtraction scheme $\lambda\mu^\varepsilon = \lambda_0$ and $D^{\mathrm{T}} = D^{\mathrm{T}}_0$, whereas $D^{\mathrm{L}}_0 = ZD^{\mathrm{L}}$.

The only superficially divergent one-particle irreducible graph corresponds to the expression

$$\begin{aligned}\Sigma_1 &= -k^2\lambda\mu^\varepsilon\int\frac{\mathrm{d}p}{(2\pi)^{d-1}}\,\frac{1}{s + D^{\mathrm{L}}k^2 + D^{\mathrm{T}}p^2}\\ &= -\frac{k^2\lambda\mu^\varepsilon\Gamma(\varepsilon/2)}{(4\pi)^{1-\varepsilon/2}(D^{\mathrm{T}})^{1-\varepsilon/2}(s + D^{\mathrm{L}}k^2)^{\varepsilon/2}}\end{aligned} \tag{6}$$

where Γ is the gamma function. Due to superrenormalizability of the model, the one-loop expressions for the renormalization constant Z, the anomalous dimension γ of the longitudinal diffusion coefficient and the beta function extracted from (6), are perturbatively *exact* in the minimal subtraction scheme. It should be noted that the asymptotic behaviour of the model is not determined by the coupling constant λ, but by a totally dimensionless expansion parameter u, which is dimensionless with respect to time, longitudinal and transverse coordinates separately. From inspection of the perturbation expansion, we infer the following expression for this parameter: $u = \lambda/D^{\mathrm{T}}D^{\mathrm{L}}$. From (6) we obtain

$$Z = 1 - \frac{u}{2\pi\varepsilon} \tag{7}$$

which yields for the anomalous dimension of the longitudinal diffusion coefficient the expression

$$\gamma(u) \equiv \mu\frac{\mathrm{d}\ln D^{\mathrm{L}}}{\mathrm{d}\mu}\bigg|_0 = -\mu\frac{\mathrm{d}\ln Z}{\mathrm{d}\mu}\bigg|_0 = -\frac{u}{2\pi} \tag{8}$$

where the subscript denotes that the derivative is taken at fixed values of the bare parameters D_0^{T}, D_0^{L}, and λ_0. The asymptotic behaviour of the model is determined by the beta function

$$\beta(u) \equiv \mu \frac{\mathrm{d}u}{\mathrm{d}\mu}\bigg|_0 = u\left(-\varepsilon + \frac{u}{2\pi}\right). \tag{9}$$

Dimensional analysis yields for the renormalized Green function G the relation

$$G(t, x, y; \mu, D^{\mathrm{L}}, D^{\mathrm{T}}, u) = \frac{R(x/\sqrt{D^{\mathrm{L}}t}, y\mu, t\mu^2 D^{\mathrm{T}}, u)}{(D^{\mathrm{L}})^{1/2}(D^{\mathrm{T}})^{(d-1)/2}t^{d/2}}. \tag{10}$$

Together with the basic renormalization group (RG) equation

$$\left[\mu\frac{\partial}{\partial\mu} + \gamma(u)D^{\mathrm{L}}\frac{\partial}{\partial D^{\mathrm{L}}} + \beta(u)\frac{\partial}{\partial u}\right]G = 0$$

which expresses the independence of the Green function G_0 of the arbitrary scaling parameter μ, the relation (10) leads to the equation

$$\left[2t\frac{\partial}{\partial t} + \left(1 - \frac{\gamma(u)}{2}\right)x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \beta(u)\frac{\partial}{\partial u} + d - \frac{\gamma(u)}{2}\right]G = 0 \tag{11}$$

from which the asymptotic behaviour of the Green function G may be inferred.

If the running coupling constant is considered as a function of time, then the exact expressions (8) and (9) yield the solution of the equation (11) in a closed form:

$$G(t, x, y; \mu, D^{\mathrm{L}}, D^{\mathrm{T}}, u) = \frac{G(1, \bar{x}, yt^{-1/2}; \mu, D^{\mathrm{L}}, D^{\mathrm{T}}, \bar{u})}{t^{d/2+\varepsilon/4}[u/2\pi\varepsilon + (1 - u/2\pi\varepsilon)t^{-\varepsilon/2}]^{1/2}} \tag{12}$$

where $\bar{x}$ and $\bar{u}$ are the first integrals of equation (11):

$$\bar{x} = \left(\frac{x}{t^{1/2+\varepsilon/4}}\right)\sqrt{\frac{\bar{u}}{u}} \qquad \bar{u} = \frac{2\pi\varepsilon}{1 - [1 - 2\pi\varepsilon/u]t^{-\varepsilon/2}}. \tag{13}$$

The beta function (9) has two fixed points: the Gaussian fixed point $u_G^* = 0$ and the non-trivial fixed point $u^* = 2\pi\varepsilon$, of which the former is infrared stable for $d > 3$, and the latter for $d < 3$, as may be seen from (13), where $\bar{u} \to 0$ in the limit $t \to \infty$, when $\varepsilon = 3 - d < 0$, whereas $\bar{u} \to 2\pi\varepsilon$, when $\varepsilon > 0$. Therefore, the asymptotic behaviour above three dimensions corresponds to the usual diffusion, whereas below three dimensions anomalous behaviour governed by the non-trivial fixed point u^* occurs. At three dimensions logarithmic corrections to the usual diffusion result.

From the relations (12) and (13) it follows that in the transverse directions the diffusion, as described by the the long-time behaviour of the mean-square displacement, is normal, whereas in the longitudinal direction below three dimensions the behaviour of the mean-square displacement is superdiffusive:

$$\langle\overline{x^2(t)}\rangle \sim t^{1+\varepsilon/2} = t^{(5-d)/2}. \tag{14}$$

Here, the bar denotes the thermal average, and the brackets the average over the random drift. The relation (14) determines the value of the exponent ν, defined by

$\langle\overline{x^2(t)}\rangle \sim t^{2\nu}$, exactly in the ε-expansion $\nu = (5-d)/4$. At three dimensions the mean-square displacement grows as

$$\langle\overline{x^2(t)}\rangle \sim t \ln t \,. \tag{15}$$

The relation (14) generalizes the two-dimensional result of [3,4] to arbitrary dimensionality, and the logarithmic correction in the relation (15) is also in agreement with the earlier heuristic result [4]. The scaling form of the longitudinal distribution of displacements follows from (12) upon integration over the transverse coordinates

$$G_L(t,x) \equiv \int \mathrm{d}y\, G(t,x,y) \sim \frac{f(x/t^{(5-d)/4})}{t^{(5-d)/4}} \qquad t \to \infty \quad d < 3\,.$$

Unfortunately, the RG argument does not allow to obtain information of the form of the scaling function f, nor does the perturbation expansion of the Green function G lead to any definitive conclusions about the asymptotic behaviour of the function f; therefore in this respect one has to rely on the previous heuristic arguments and numerical results [4].

The generalization of the problem (1) and (2) to the case of a velocity field with a ($d' \leqslant d$)-dimensional 'Manhattan grid' structure [4,5] is described by the equation

$$\begin{aligned}\Bigg\{\frac{\partial}{\partial t} - D_0^{\mathrm{T}} \sum_{m=1}^{d-d'} \frac{\partial^2}{\partial y_m^2} - \sum_{n=1}^{d'} \Bigg[D_{0n} \frac{\partial^2}{\partial x_n^2} \\ - \psi_n(x_1,\ldots,\hat{x}_n,\ldots,x_{d'},y_1,\ldots,y_{d-d'}) \frac{\partial}{\partial x_n}\Bigg]\Bigg\} P(t,\boldsymbol{x}) = 0\end{aligned} \tag{16}$$

where the coordinates in the unbiased ('transverse') directions have been denoted by $\boldsymbol{y}$, D_0^{T} is the bare transverse diffusion coefficient, and D_{0n} are the bare diffusion coefficients in the biased directions. The hat over the argument of a ψ function means that the function does not depend on that coordinate. The random fields ψ_n have Gaussian distributions with zero mean and the correlation functions

$$\langle\psi_n(\boldsymbol{x},\boldsymbol{y})\psi_m(\boldsymbol{x}',\boldsymbol{y}')\rangle = \delta_{mn}\lambda_{0n}\delta(\boldsymbol{y}-\boldsymbol{y}') \prod_{l\neq n}^{d'} \delta(x_l - x_l')\,. \tag{17}$$

The field theory corresponding to (16) and (17) is determined by the action

$$\begin{aligned}S = -\sum_{n=1}^{d'} \int \Bigg(\prod_{m\neq n}^{d'} \mathrm{d}x_m \Bigg) \mathrm{d}\boldsymbol{y}\, \frac{1}{2\lambda_{0n}} \psi_n^2 \\ + \int \mathrm{d}t\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\, \tilde{\varphi} \Bigg[\frac{\partial}{\partial t} - D_0^{\mathrm{T}} \sum_{m=1}^{d-d'} \frac{\partial^2}{\partial y_m^2} - \sum_{n=1}^{d'} \Bigg(D_{0n} \frac{\partial^2}{\partial x_n^2} - \psi_n \frac{\partial}{\partial x_n} \Bigg) \Bigg] \varphi\,.\end{aligned} \tag{18}$$

It is convenient to carry out dimensional analysis by introducing separately dimensions with respect to each biased coordinate, and defining the total scaling dimension of a variable v by $d_v = 2d_v^{\mathrm{s}} + d_v^{\mathrm{T}} + \sum_{n=1}^{d'} d_v^n$. The total dimensions of the bare coupling constants λ_0 are $d_{\lambda_{0n}} = 3 - d$, i.e. the critical dimension of this model is $d_c = 3$, too. Power counting shows that this field theory is not superrenormalizable: there is an infinite set of different superficially divergent graphs in its perturbation expansion. However, as a result of the vertex momentum factorization, only the diffusion constants D_{0n} are renormalized (apart from finite renormalization), and the renormalized action may be obtained from (18) by the substitutions $\lambda_{0n} = \lambda_n \mu^\varepsilon$, $D_0^{\rm T} = D^{\rm T}$, and $D_{0n} = D_n Z_n$, where λ_n, $D^{\rm T}$, and D_n are the renormalized parameters of the model, and Z_n are the renormalization constants of the corresponding diffusion coefficients. The totally dimensionless expansion parameters are $u_n = \lambda_n / \left(D_n \sqrt{(D^{\rm T})^{d-d'} \prod_{m\neq n}^{d'} D_m}\right)$ which correspond to the following choice of the dimensionality of the scaling parameter μ: $d_\mu^{\rm s} = 1/2$, $d_\mu^n = d_\mu^{\rm T} = 0$.

In the one-loop approximation the values of the renormalization constants are obtained from the same integral (6) as in the case of unidirectional convection and are equal to (in the minimal subtraction scheme)

$$Z_n = 1 - \frac{u_n}{2\pi\varepsilon} + \cdots . \tag{19}$$

The interaction vertices in the action (18) are unrenormalized, therefore

$$\beta_n \equiv \mu \frac{\mathrm{d}u_n}{\mathrm{d}\mu}\bigg|_0 = u_n \left(-\varepsilon - \gamma_n - \frac{1}{2} \sum_{m\neq n}^{d'} \gamma_m \right) \tag{20}$$

where γ_n are the anomalous dimensions of the diffusion coefficients defined as

$$\gamma_n(u_1, \ldots, u_d) \equiv \mu \frac{\mathrm{d}\ln D_n}{\mathrm{d}\mu}\bigg|_0 = -\mu \frac{\mathrm{d}\ln Z_n}{\mathrm{d}\mu}\bigg|_0 . \tag{21}$$

At one-loop order we obtain from (19–21)

$$\beta_n = u_n \left(-\varepsilon + \frac{u_n}{2\pi} + \frac{1}{4\pi} \sum_{m\neq n}^{d'} u_m \right) .$$

The fixed-point equations $\beta_n = 0$ have several non-trivial solutions, i.e. solutions in which at least one of the coupling constants u_n does not vanish. It can be readily seen that there is a unique non-trivial fixed point with *all* non-vanishing components: $u_n^* = 4\pi\varepsilon/(d'+1)$; there are d' fixed points such that one of the coupling constants is zero and the others not: $u_n^* = 0$, $u_m^* = 4\pi\varepsilon/d' \quad m \neq n$; there are $d'(d'-1)/2$ fixed points with two vanishing coupling constants the others being equal to $4\pi\varepsilon/(d'-1)$, etc. Finally, there are d' non-trivial solutions with only one non-vanishing coupling constant equal to $u_n^* = 2\pi\varepsilon$. Formally, the stability of these fixed points is determined by the eigenvalues of the matrix $\partial\beta_n/\partial u_m$ at the fixed point, and it turns out that this matrix is positively determined below the critical dimension $d < 3$ only in the case of the fixed point with all non-vanishing coupling constants, which means that in the general case this isotropic fixed point governs the long-distance and long-time asymptotic behaviour of the model, regardless to the initial values of the diffusion constants and the coupling constants in the biased directions (provided the latter do not vanish). However, if we put some of the coupling constants equal to zero from the very beginning in the action (18), then the corresponding vertex factors are not generated in the course of renormalization. The reason is that the derivatives at the vertices, to which the external φ and $\tilde{\varphi}$ legs are attached, always yield factors with momenta corresponding to coupling constants already present, and thus leave no possibility for a new vertex with linear momentum dependence to appear. In this case the fixed point with the chosen coupling constants vanishing and the rest non-zero turns out to be infrared stable!

The absence of the (divergent) vertex renormalization in this model has the important consequence that the fixed point equation of the RG determines the anomalous dimensions γ_n of the diffusion coefficients *exactly* in the ε-expansion. This phenomenon is familiar from earlier studies of diffusion in divergenceless random fields [2,6]. From (21) we thus obtain

$$\gamma_n(u_1^*, \ldots, u_{d'}^*) \equiv \gamma = -\frac{2\varepsilon}{d'+1} = \frac{2(d-3)}{d'+1} . \tag{22}$$

With the aid of dimensional analysis the RG equation may be written in the form

$$\left[2t\frac{\partial}{\partial t} + \sum_{n=1}^{d'}\left(1-\frac{\gamma_n}{2}\right)x_n\frac{\partial}{\partial x_n} + \sum_{n=1}^{d'}\beta_n\frac{\partial}{\partial u_n} + d - \frac{1}{2}\sum_{n=1}^{d'}\gamma_n\right]G = 0 . \tag{23}$$

The leading asymptotic behaviour at long times and large distances is given by the solution of the equation (23) at the infrared stable fixed point of the RG, at which

$$G(t, x_n, y_m; \mu, D_n, D^{\mathrm{T}}, u_n^*) = \frac{G(1, x_n/t^{(4+d'-d)/2(d'+1)}, y_m/\sqrt{t}; \mu, D_n, D^{\mathrm{T}}, u_n^*)}{t^{(3d'+d)/(d'+1)}} \tag{24}$$

where $u_n^* = 4\pi\varepsilon/(d'+1)$. From this expression we obtain the perturbatively *exact* value of the exponent $\nu = (4+d'-d)/2(d'+1)$ for $d < 3$ in the biased directions. For the isotropic case $d' = d$ this is in accord with the earlier heuristic result [4]. For the mean-square displacement we therefore obtain

$$\langle\overline{x^2(t)}\rangle \sim t^{(4+d'-d)/(d'+1)} \qquad t \to \infty \tag{25}$$

below three dimensions. Logarithmic corrections of the form

$$\langle\overline{x^2(t)}\rangle \sim t(\ln t)^{2/(d'+1)} \qquad t \to \infty \tag{26}$$

occur at three dimensions, and the diffusion is normal above three dimensions. The results for the case of random unidirectional convection (equations (14) and (15)) are recovered by the substitution $d' = 1$ in (25) and (26).

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